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MODULI SPACES FOR COVERS OF THE RIEMANN SPHERE*

BY

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ABSTRACT

Moduli spaces for covers of the Riemann sphere have been constructed in a joint work with M. Fried [FV1]. They were used to realize groups as Galois groups [FV1], [Vö1], and to determine the absolute Galois group of large fields [FV2]. Here we simplify and extend the construction of these moduli spaces.

Introduction

Moduli spaces for covers of the Riemann sphere were first studied by Hurwitz [Hu] in the case of simple covers. Fulton [Fu] introduced the algebraic structure on these moduli spaces for simple covers. The theory for arbitrary covers originated in [Fr], and was further developed in joint work of M. Fried and the author [FV1]. It has been applied to solve embedding problems over PAC-fields [FV2] (leading to a structural result about the absolute Galois group of the rationals); as further application, many general linear and unitary groups were realized as Galois groups over the rationals, see [Vö1], [Vö2], [Vö3].

For further Galois realizations, a certain extension of the main result of [FV1] is needed, where the Hurwitz braid group is replaced by the original Artin braid group. This could have been derived in the context of the [FV1] paper, but when

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working on this, the author found a more direct description of the algebraic structure on the moduli spaces. This new approach does not rely on the generalized Riemann existence theorem, but needs only the 1-dimensional version (the classical Riemann existence theorem), and some basic results from the theory of several complex variables (like Riemann's extension theorem for normal complex spaces). Note that also the parallel field-theoretic work of Matzat [Ma1], [Ma2] is based on the generalized existence theorem (more precisely, on the identification between the algebraic fundamental group and the completion of the topological one).

The moduli spaces $\mathcal{H}_r^{\text{in}}(G)$ arise firstly as topological coverings of the complement $\mathcal{U}_r$ of the discriminant locus in projective r-space. Thus these spaces are determined topologically by a permutation representation of the fundamental group of the base space $\mathcal{U}_r$. This permutation representation has been described explicitly in [FV1]. In §3 we give the corresponding result for the new spaces $\mathcal{H}_r(G)$ constructed in the present paper.

In §1 we begin again with the topological definition. A new feature is here that we give a direct topological definition of the family $\mathcal{T}_r(G)$ of all Galois covers of $\mathbb{P}^1$ with Galois group G and r branch points. In §2 it is shown that $\mathcal{H}_r^{\text{in}}(G)$ and $\mathcal{T}_r(G)$ are the union of closed and open subspaces each of which can be identified with the normalization of a suitable open subset of projective r-space in a certain field associated to a generic cover in the family. This yields the algebraic structure on the moduli spaces.

In §3 we state the corollaries (needed for further Galois realizations in [Vö3]) that gave the original motivation for this paper. Recall that the Artin braid group $\mathcal{B}_r$ is the fundamental group of the complement $\mathcal{O}_r$ of the discriminant locus in affine r-space. We show that the topological covering $\mathcal{H}_r(G)$ of $\mathcal{O}_r$ associated to the braiding action of $\mathcal{B}_r$ on certain generating systems of a finite group G has a structure of $\mathbb{Q}$-variety. Further, the group $\text{Aut}(G)$ acts on this variety in a natural way. The corresponding fact was known for the braiding action on $\text{Inn}(G)$-classes of generating systems (by [FV1]), but not for the action on the generating systems themselves.

In §3B we study certain subspaces of $\mathcal{H}_r(G)$ associated to r-tuples of conjugacy classes of G. In particular, we determine the field of definition of these subspaces. This is again parallel to the corresponding results for $\mathcal{H}_r^{\text{in}}(G)$ in [FV1].

NOTATIONS. All occurring fields are of characteristic 0. The algebraic closure of a field k is denoted by $\bar{k}$, and we let $\mathbf{G}_k = G(\bar{k}/k)$ denote the absolute Galois group of k. A field L is said to be *regular* over a subfield k if k is algebraically closed in L.

The field of rationals (resp., complexes) is denoted by $\mathbb{Q}$ (resp., $\mathbb{C}$). We let $\mathbb{P}^1$ denote the Riemann sphere $\mathbb{C} \cup \{\infty\}$, viewed according to context as a Riemann surface or as an algebraic curve defined over $\mathbb{Q}$ (in the natural way). The fundamental group of a topological space Y, based at $y \in Y$, is denoted $\pi_1(Y, y)$. By "covering" we mean an unramified topological covering (in the sense of covering space theory) of not necessarily connected spaces. The word "cover", on the other hand, is reserved for connected branched covers of the Riemann sphere.

The algebraic varieties we consider are all defined over subfields of $\mathbb{C}$. We identify a variety with its set of complex points.

Throughout the paper we fix the following: a finite group G, and an integer $r \geq 3$.

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1. Topological definition of the moduli spaces

1.1 Embed the affine r-space $\mathbf{A}^r$ into the projective r-space $\mathbb{P}^r$ by regarding $\mathbf{A}^r$ as the space of monic complex polynomials of degree r, and $\mathbb{P}^r$ as the space of all nonzero complex polynomials of degree at most r up to multiplication by a nonzero constant. Consider the classical discriminant locus in $\mathbf{A}^r$, corresponding to the polynomials with repeated roots, and denote its closure in $\mathbb{P}^r$ by D_r. We will work with the space $\mathcal{U}_r \overset{\text{def}}{=} \mathbb{P}^r \setminus D_r$, which we view as the space of all subsets of cardinality r of the Riemann sphere $\mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$. That is, we identify a point of $\mathcal{U}_r$ with the set of roots of a corresponding polynomial, where we count ∞ as a root if the degree of the polynomial is less than r (the degree is then necessarily $r-1$). Under this identification, the (open) subspace $\mathcal{O}_r \overset{\text{def}}{=} \mathbf{A}^r \setminus D_r$ of $\mathcal{U}_r$ becomes the space of all subsets of cardinality r of $\mathbb{C}$.

The spaces $\mathcal{U}_r$ and $\mathcal{O}_r$ have a natural structure as algebraic variety defined over $\mathbb{Q}$. For the moment we consider $\mathcal{U}_r$ only as a complex manifold. Its (complex) topology can be described as follows: For any pairwise disjoint open discs

$D_1, \ldots, D_r$ on $\mathbb{P}^1$ consider the set of all $\mathbf{a} \in \mathcal{U}_r$ with $|\mathbf{a} \cap D_i| = 1$ for $i = 1, \ldots, r$. These sets form a basis for the topology.

1.2 From now on $\varphi\colon X \to \mathbb{P}^1$ will always denote a (branched) cover of compact (connected) Riemann surfaces. Two such covers $\varphi\colon X \to \mathbb{P}^1$ and $\varphi'\colon X' \to \mathbb{P}^1$ are called equivalent if there exists an isomorphism $\delta\colon X \to X'$ with $\varphi' \circ \delta = \varphi$. We let $\mathrm{Aut}(X/\mathbb{P}^1)$ denote the group of automorphisms of the cover $\varphi\colon X \to \mathbb{P}^1$ (i.e., automorphisms δ of X with $\varphi \circ \delta = \varphi$). The cover φ is called a Galois cover if $\mathrm{Aut}(X/\mathbb{P}^1)$ is transitive on the fibers of φ. From now on we consider only Galois covers.

Let $a_1, \ldots, a_r \in \mathbb{P}^1$ be the branch points of the cover φ and set $\boldsymbol{a} = \{a_1, \ldots, a_r\}$. Then φ restricts to an (unramified) topological cover φ^0 of the punctured sphere $\mathbb{P}^1 \setminus \boldsymbol{a}$. Choose a base point a_0 on this punctured sphere. By the theory of covering spaces, the equivalence class of φ^0 corresponds to a normal subgroup U_φ of the fundamental group $\Gamma = \pi_1(\mathbb{P}^1 \setminus \boldsymbol{a}, a_0)$. In fact we have a 1-1 correspondence between the equivalence classes of Galois covers $\varphi'\colon X' \to \mathbb{P}^1$ with branch points among $a_1, \ldots, a_r$, and normal subgroups of Γ of finite index (see e.g. [Fu, 1.3]). Under this correspondence, the covers with *exactly* r branch points correspond to those subgroups of Γ that do not contain the kernel of the natural map from Γ to $\Gamma_i \overset{\text{def}}{=} \pi_1((\mathbb{P}^1 \setminus \boldsymbol{a}) \cup \{a_i\}, a_0)$, for any i.

Depending on the choice of a base point $p \in \varphi^{-1}(a_0)$, we get a surjection $\iota\colon \Gamma \to \mathrm{Aut}(X/\mathbb{P}^1)$ as follows: For each path γ representing an element of Γ, let q be the endpoint of the unique lift of γ to $X \setminus \varphi^{-1}(\boldsymbol{a})$ with initial point p; then ι sends γ to the unique element ϵ of $\mathrm{Aut}(X/\mathbb{P}^1)$ with $\epsilon(q) = p$. Varying p over $\varphi^{-1}(a_0)$ means composing ι with inner automorphisms of $\mathrm{Aut}(X/\mathbb{P}^1)$.

1.3 Let $\mathcal{H}^{\mathrm{in}} = \mathcal{H}^{\mathrm{in}}_r(G)$ be the set of equivalence classes of pairs (φ, h) where $\varphi\colon X \to \mathbb{P}^1$ is a Galois cover with r branch points, and $h\colon \mathrm{Aut}(X/\mathbb{P}^1) \to G$ is an isomorphism; two such pairs (φ, h) and $(\varphi'\colon X' \to \mathbb{P}^1,\ h')$ are called equivalent if there is an isomorphism $\delta\colon X \to X'$ over $\mathbb{P}^1$ such that $h' \circ c_\delta = h$, where $c_\delta\colon \mathrm{Aut}(X/\mathbb{P}^1) \to \mathrm{Aut}(X'/\mathbb{P}^1)$ is the isomorphism induced by δ (i.e, $c_\delta(\epsilon) = \delta \circ \epsilon \circ \delta^{-1}$). Let $|\varphi, h|$ denote the equivalence class of the pair (φ, h).

Note that points of $\mathcal{H}^{\mathrm{in}}$ can equally well be thought of as equivalence classes of triples $(\boldsymbol{a}, a_0, f)$, where $\boldsymbol{a} = \{a_1, \ldots, a_r\} \in \mathcal{U}_r$, $a_0 \in \mathbb{P}^1 \setminus \boldsymbol{a}$ and $f\colon \Gamma = \pi_1(\mathbb{P}^1 \setminus \boldsymbol{a}, a_0) \to G$ is a surjection that does not factor through the canonical map $\Gamma \to \Gamma_i$, for any i. Two such triples $(\boldsymbol{a}, a_0, f)$ and $(\tilde{\boldsymbol{a}}, \tilde{a}_0, \tilde{f})$ are called equivalent if $\boldsymbol{a} = \tilde{\boldsymbol{a}}$ and there is a path γ from a_0 to $\tilde{a}_0$ in $\mathbb{P}^1 \setminus \boldsymbol{a}$ such that the induced map $\gamma^*\colon \pi_1(\mathbb{P}^1 \setminus \boldsymbol{a}, a_0) \to \pi_1(\mathbb{P}^1 \setminus \boldsymbol{a}, \tilde{a}_0)$ satisfies $\tilde{f} \circ \gamma^* = f$.

Here is the correspondence between the above pairs and triples. In the above notation, h and f are related by $f = h \circ \iota$, and $U_\varphi = \ker(f)$ $(= \ker(\iota))$. Varying p over $\varphi^{-1}(a_0)$ means composing ι with inner automorphisms of $\mathrm{Aut}(X/\mathbb{P}^1)$. Therefore h and f determine each other up to inner automorphisms of G, which is compatible with the equivalence of pairs (resp., triples).

1.4 Let $\mathcal{T}_r(G)$ be the set of equivalence classes of triples (φ, h, p) where $\varphi\colon X \to \mathbb{P}^1$ is a Galois cover with r branch points, $h\colon \mathrm{Aut}(X/\mathbb{P}^1) \to G$ is an isomorphism, and p is a point of X such that $\varphi(p)$ is not a branch point. Two such triples (φ, h, p) and (φ', h', p') are called equivalent if there is an isomorphism $\delta\colon X \to X'$ over $\mathbb{P}^1$ such that $\delta(p) = p'$ and $h' \circ c_\delta = h$, where $c_\delta\colon \mathrm{Aut}(X/\mathbb{P}^1) \to \mathrm{Aut}(X'/\mathbb{P}^1)$ is the isomorphism induced by δ. Let $|\varphi, h, p|$ denote the equivalence class of the triple (φ, h, p).

Note that points of $\mathcal{T}_r(G)$ can equally well be thought of as triples $(\boldsymbol{a}, a_0, f)$ as in 1.3 (but now with no equivalences between them). Given (φ, h, p) as above, let $a_1, ..., a_r$ be the branch points of φ, set $a_0 = \varphi(p)$ and $f = h \circ \iota$; thereby, $\iota\colon \Gamma \to \mathrm{Aut}(X/\mathbb{P}^1)$ is the surjection corresponding to the base point $p \in \varphi^{-1}(a_0)$ (see 1.2).

Let $\Psi^{\text{in}}\colon \mathcal{H}^{\text{in}} \to \mathcal{U}_r$ be the map sending $|\varphi, h|$ to the set of branch points of φ. Let $\lambda\colon \mathcal{T}_r(G) \to \mathcal{H}^{\text{in}}$ and $\mu\colon \mathcal{T}_r(G) \to \mathbb{P}^1$ be the maps sending $|\varphi, h, p|$ to $|\varphi, h|$ and $\varphi(p)$, respectively. Let $\Lambda_r\colon \mathcal{T}_r(G) \to \mathcal{H}^{\text{in}} \times \mathbb{P}^1$ be the map $\lambda \times \mu$, and let $\Phi^{(r)}\colon \mathcal{T}_r(G) \to \mathcal{U}_r \times \mathbb{P}^1$ be Λ_r composed with $\Psi^{\text{in}} \times \mathrm{id}$.

Let $\mathcal{S}$ be the image of Λ_r (the set of all $(\boldsymbol{p}, a) \in \mathcal{H}^{\text{in}} \times \mathbb{P}^1$ with $a \notin \Psi^{\text{in}}(\boldsymbol{p})$), and let $\mathcal{U}(r+1)$ be the image of $\Phi^{(r)}$ (the set of all $(\boldsymbol{a}, a) \in \mathcal{U}_r \times \mathbb{P}^1$ with $a \notin \boldsymbol{a}$).

1.5 The sets $\mathcal{T}_r(G)$ and $\mathcal{H}^{\text{in}}$ carry a natural topology such that $\Phi^{(r)}\colon \mathcal{T}_r(G) \to \mathcal{U}(r+1)$ and $\Psi^{\text{in}}\colon \mathcal{H}^{\text{in}} \to \mathcal{U}_r$ become (unramified) coverings. To specify a neighborhood $\mathcal{M}(D_0, ..., D_r)$ (resp., $\mathcal{N}(D_1, ..., D_r)$) of the point of $\mathcal{T}_r(G)$ (resp., $\mathcal{H}^{\text{in}}$) represented by the triple $(\{a_1, ..., a_r\}, a_0, f)$, choose pairwise disjoint discs $D_0, \ldots, D_r$ around the points $a_0, \ldots, a_r$. Then the set $\mathcal{M}(D_0, ..., D_r)$ (resp., $\mathcal{N}(D_1, ..., D_r)$) consists of all points represented by the triples $(\{\tilde{a}_1, ..., \tilde{a}_r\}, \tilde{a}_0, \tilde{f})$, such that there is exactly one $\tilde{a}_i$ in each D_i, and $\tilde{f}$ is the composition of the canonical isomorphisms

$$(1) \qquad \pi_1(\mathbb{P}^1 \setminus \tilde{\boldsymbol{a}}, \tilde{a}_0) \cong \pi_1(\mathbb{P}^1 \setminus (D_1 \cup \cdots \cup D_r), a_0) \cong \pi_1(\mathbb{P}^1 \setminus \boldsymbol{a}, a_0)$$

with f. (Note these isomorphisms are canonical because D_0 is simply-connected). These sets form a basis for the neighborhoods of the given point of $\mathcal{T}_r(G)$ (resp., $\mathcal{H}^{\text{in}}$).

Now one checks easily that actually the maps $\mathcal{H}^{\text{in}} \to \mathcal{U}_r$ and $\mathcal{T}_r(G) \to \mathcal{U}(r+1)$ are coverings: Consider the inverse image of the open subset of $\mathcal{U}_r$ (resp., $\mathcal{U}(r+1)$) given by $D_1, \ldots, D_r$ (resp., $D_0, \ldots, D_r$) as in 1.1. This inverse image is a disjoint union of copies of $D_1 \times \cdots \times D_r$ (resp., $D_0 \times \cdots \times D_r$) corresponding to the various surjections $\pi_1(\mathbb{P}^1 \setminus (D_1 \cup \cdots \cup D_r), a_0) \to G$ (modulo $\mathrm{Inn}(G)$ in the case of $\mathcal{H}^{\text{in}}$). Also the map $\Lambda_r\colon \mathcal{T}_r(G) \to \mathcal{S}$ is a covering if we equip $\mathcal{S}$ with the topology of an open subspace of $\mathcal{H}^{\text{in}} \times \mathbb{P}^1$. Through these coverings the spaces $\mathcal{H}^{\text{in}}$ and $\mathcal{T}_r(G)$ inherit a structure of complex manifold.

For each automorphism A of G, define the maps $\epsilon_A\colon \mathcal{T}_r(G) \to \mathcal{T}_r(G)$ and $\delta_A\colon \mathcal{H}^{\text{in}} \to \mathcal{H}^{\text{in}}$ by sending $|\varphi, h, p|$ and $|\varphi, h|$ to $|\varphi, Ah, p|$ and $|\varphi, Ah|$, respectively. One checks that these maps are continuous, hence they are automorphisms of the coverings $\Phi^{(r)}$ and Ψ^{in}, respectively. Those maps ϵ_A with $A \in \mathrm{Inn}(G)$ are even automorphisms of the covering $\Lambda_r\colon \mathcal{T}_r(G) \to \mathcal{S}$, and the induced map $\mathcal{T}_r(G)/\,\mathrm{Inn}(G) \to \mathcal{S}$ is homeomorphic.

1.6 Here is our main result (to be proved in §2):

MAIN THEOREM: *Let G be a finite group that can be generated by $r-1$ elements. Then the spaces $\mathcal{H}_r^{\text{in}}(G)$ and $\mathcal{T}_r(G)$ have a unique structure as (reducible) algebraic varieties defined over $\mathbb{Q}$ (compatible with their analytic structure) so that the maps*

$$\Psi^{\text{in}}\colon \mathcal{H}_r^{\text{in}}(G) \to \mathcal{U}_r \quad \text{and} \quad \mathcal{T}_r(G) \xrightarrow{\Lambda_r} \mathcal{H}_r^{\text{in}}(G) \times \mathbb{P}^1 \to \mathcal{U}_r \times \mathbb{P}^1$$

and all ϵ_A, δ_A ($A \in Aut(G)$) are algebraic morphisms defined over $\mathbb{Q}$, and the following holds: For the action of the automorphisms β of $\mathbb{C}$ on the complex points of $\mathcal{H}_r^{\text{in}}(G)$ and $\mathcal{T}_r(G)$, we have

$$|\varphi, h|^\beta = |\varphi^\beta, h\beta^{-1}| \quad \text{and} \quad |\varphi, h, p|^\beta = |\varphi^\beta, h\beta^{-1}, p^\beta|.$$

The meaning of the right hand sides of these formulas will be explained in 2.1 and 2.2. The space $\mathcal{U}_r$ is considered as variety defined over $\mathbb{Q}$ in the natural way (see 1.1.). Note that the space $\mathcal{H}_r^{\text{in}}(G)$ is non-empty if and only if G can be generated by $r-1$ elements. (This follows from Riemann's existence theorem, cf. [FV1, 1.3]).

The result for $\mathcal{H}_r^{\text{in}}(G)$ is essentially equivalent to Theorem 1 of [FV1]. (The results about the spaces $\mathcal{H}_r^{\text{ab}}(G, U)$ follow immediately from those for $\mathcal{H}_r^{\text{in}}(G)$). The result for $\mathcal{T}_r(G)$ is not explicitly known (and it would take some work to derive it from [FV1]).

2. The algebraic structure of the moduli spaces

2.1 For each cover $\varphi: X \to \mathbb{P}^1$ (of connected compact Riemann surfaces) the space X has a unique structure as algebraic variety defined over $\mathbb{C}$ (compatible with its analytic structure) such that φ becomes an algebraic morphism (Riemann's existence theorem). Thus for each automorphism β of $\mathbb{C}$, we can form the cover $\varphi^\beta: X^\beta \to \mathbb{P}^1$ obtained from $\varphi: X \to \mathbb{P}^1$ through base change with β.

We say that φ can be defined over some subfield k of $\mathbb{C}$ if X can be given a structure of variety defined over k such that φ becomes a morphism defined over k.

Let $\boldsymbol{a} = \{a_1, \ldots, a_r\} \in \mathcal{U}_r$ be the set of branch points of φ, and $k_0 = \mathbb{Q}(\boldsymbol{a})$; thus k_0 is the field generated by the coefficients of the polynomial $\prod(x - a_i)$, where the product is over those $i = 1, \ldots, r$ with $a_i \neq \infty$. Now let k be any subfield of $\mathbb{C}$ over which φ can be defined. Then the branch points $a_1, \ldots, a_r$ are algebraic over k, and the absolute Galois group $\mathbf{G}_k$ permutes $a_1, \ldots, a_r$. Therefore k_0 is contained in k. Further, φ can actually be defined over some finite algebraic extension of k_0 (see [FV1, 1.5]). We want to determine a (minimal) field over which φ together with all its automorphisms can be defined. (From now on φ is again a Galois cover).

2.2 The group $\text{Aut}(\mathbb{C})$ (the automorphism group of $\mathbb{C}$) acts as follows on the points $\boldsymbol{p} = |\varphi, h|$ of $\mathcal{H}^{\text{in}}$:

$$\boldsymbol{p}^\beta = |\varphi^\beta, h\beta^{-1}|$$

for all $\beta \in \text{Aut}(\mathbb{C})$. Thereby $h\beta^{-1}$: $\text{Aut}(X^\beta/\mathbb{P}^1) \to G$ is the isomorphism sending A^β to $h(A)$ for every $A \in \text{Aut}(X/\mathbb{P}^1)$.

Clearly, if k is a subfield of $\mathbb{C}$ over which φ together with all its automorphisms can be defined, then all $\beta \in \text{Aut}(\mathbb{C}/k)$ fix the point $\boldsymbol{p} = |\varphi, h|$ of $\mathcal{H}^{\text{in}}$. For the converse we have to assume that the group $G \cong \text{Aut}(X/\mathbb{P}^1)$ has trivial center. Then actually φ together with all its automorphisms can be defined over the fixed field of the group of all $\beta \in \text{Aut}(\mathbb{C})$ with $\boldsymbol{p}^\beta = \boldsymbol{p}$. This follows by a simple application of Weil's cocycle criterion (see the proof of [FV1, Cor. 1]).

2.3 Now let φ_0: $X_0 \to \mathbb{P}^1$ be a Galois cover with group isomorphic to G, and with r branch points $t_1, \ldots, t_r \in \mathbb{C}$ that are algebraically independent over $\mathbb{Q}$. Set $\mathbf{t} = \{t_1, \ldots, t_r\} \in \mathcal{U}_r$, and let $x_1, \ldots, x_r$ be the elementary symmetric functions in $t_1, \ldots, t_r$. Then $k_0 \stackrel{\text{def}}{=} \mathbb{Q}(\mathbf{t}) = \mathbb{Q}(x_1, \ldots, x_r)$. By 2.1. there is a finite algebraic extension k of k_0 (inside $\mathbb{C}$) such that φ_0 together with all its automorphisms can be defined over k. Let $L/k(x)$ be the corresponding function field extension, where x is the identity function on $\mathbb{P}^1$. Then L is regular over k, and $L/k(x)$ is Galois with group isomorphic to G.

Lemma: *Let R_1 be a subring of $\mathbb{Q}(x_1, \ldots, x_r)$ containing $\mathbb{Q}[x_1, \ldots, x_r]$, and let R be the integral closure of R_1 in k. There is $u \neq 0$ in R such that for every $\mathbb{Q}$-algebra homomorphism λ: $R \to \mathbb{C}$ with $\lambda(u) \neq 0$ the following holds:*

Let S be the integral closure of $R[x]$ in L, and let $\bar{\lambda}$ be a homomorphism from S into the algebraic closure of $\mathbb{C}(x)$ that extends λ and fixes x. Let k' and L' be the fields generated by the $\bar{\lambda}$-images of R and S, respectively. Then $L'/k'(x)$ is Galois with Galois group canonically isomorphic to $G(L/k(x))$, and $L'/k'(x)$ is ramified at exactly r (distinct) places $t'_1, \ldots, t'_r$; the elementary symmetric functions in $t'_1, \ldots, t'_r$ are the $\lambda(x_1), \ldots, \lambda(x_r)$. Further, L' is regular over k'.

Proof: Similar as for [FV1, Lemma 0]. Use additionally that we can write $L = k(x, y_1) = k(x, y_2)$, with minimal equations $f_1(x, y_1) = 0 = f_2(x, y_2)$, such that the branch points of $L/k(x)$ are exactly the common zeroes of the discriminant of f_1 and of f_2 (where the f_i are viewed as polynomials in y, and the discriminant is then a polynomial in x). ∎

2.4 Embed the space $\mathcal{U}_r$ into projective space $\mathbb{P}^r$ as in 1.1. Let $X_1, \ldots, X_r$ be the coordinate functions on $\mathbf{A}^r$ (where $\mathbf{A}^r$ is embedded in $\mathbb{P}^r$ as in 1.1). Then evaluation of functions at the point $\mathbf{t} \in \mathcal{U}_r$ yields an isomorphism $\mathbb{Q}(\mathcal{U}_r) = \mathbb{Q}(X_1, \ldots, X_r) \to \mathbb{Q}(x_1, \ldots, x_r)$. The field extensions

$$k \geq \mathbb{Q}(x_1, \ldots, x_r)$$

and

$$L \geq k(x) \geq \mathbb{Q}(x_1, \ldots, x_r, x)$$

give rise to sequences of normal varieties defined and irreducible over $\mathbb{Q}$:

$$\mathcal{H} \xrightarrow{\Psi} \mathcal{U}_r$$

and

$$\mathcal{T} \xrightarrow{\Lambda} \mathcal{H} \times \mathbb{P}^1 \rightarrow \mathcal{U}_r \times \mathbb{P}^1 \tag{2}$$

(by taking normalizations in the respective function fields). Let $\Phi: \mathcal{T} \to \mathcal{U}_r \times \mathbb{P}^1$ denote the map in (2).

The group G is isomorphic to the group of all algebraic automorphisms α, defined over $\mathbb{Q}$, of the map $\mathcal{T} \to \mathcal{H} \times \mathbb{P}^1$ (i.e., automorphisms α of $\mathcal{T}$ with $\Lambda \circ \alpha = \Lambda$). We fix an isomorphism between G and this automorphism group.

There is $\boldsymbol{p}_0 \in \mathcal{H}$ lying over the point $\mathbf{t}$ of $\mathcal{U}_r$ such that evaluation of functions at this point yields an isomorphism $\mathbb{Q}(\mathcal{H}) \to k$ (extending the map $\mathbb{Q}(\mathcal{U}_r) \to \mathbb{Q}(x_1, ..., x_r)$). This also yields an isomorphism $\mathbb{Q}(\mathcal{H} \times \mathbb{P}^1) \to k(x)$, which extends to an isomorphism $\mathbb{Q}(\mathcal{T}) \to L$.

2.5 Let $\pi: \mathcal{H} \times \mathbb{P}^1 \to \mathcal{H}$ be projection. For each $\boldsymbol{p} \in \mathcal{H}$ let $\mathcal{T}_{\boldsymbol{p}} = (\pi \circ \Lambda)^{-1}(\boldsymbol{p})$ denote its fiber in $\mathcal{T}$; it has a natural map $\varphi_{\boldsymbol{p}}$: $\mathcal{T}_{\boldsymbol{p}} \to \mathbb{P}^1$. Since $\mathcal{T}$ is normal, its singular locus has codimension at least 2. Hence the image of this singular locus in $\mathcal{H}$ is contained in a proper subvariety (as $\dim(\mathcal{H}) = \dim(\mathcal{T}) - 1$). Thus over a Zariski-open subset of $\mathcal{H}$, there are no singularities of $\mathcal{T}$. Then there is also a Zariski-open subset of $\mathcal{H}$ such that the fibers $\mathcal{T}_{\boldsymbol{p}}$ are non-singular over this subset (by "generic smoothness" [Ha,III, Cor.10.7]). Now it follows from Lemma 2.3 that this subset contains another Zariski-open subset $\mathcal{H}_0$ of $\mathcal{H}$ defined over $\mathbb{Q}$ with the following property: For each $\boldsymbol{p} \in \mathcal{H}_0$ the map $\varphi_{\boldsymbol{p}}$: $\mathcal{T}_{\boldsymbol{p}} \to \mathbb{P}^1$ is a (connected) Galois cover of $\mathbb{P}^1$, ramified exactly at the points in $\Psi(\boldsymbol{p})$. The group $\mathrm{Aut}(\mathcal{T}_{\boldsymbol{p}}/\mathbb{P}^1)$ is isomorphic to G, via restriction of the action of G on $\mathcal{T}$; denote this isomorphism by $h_{\boldsymbol{p}}$: $\mathrm{Aut}(\mathcal{T}_{\boldsymbol{p}}/\mathbb{P}^1) \to G$.

Removing the branch locus (= the image of the zero set of the Jacobian determinant) of Ψ, we can further assume that $\Psi: \mathcal{H}_0 \to \Psi(\mathcal{H}_0)$ is an unramified covering (in the complex topology), where $\Psi(\mathcal{H}_0)$ is a Zariski-open subset of $\mathcal{U}_r$. Set

$$\mathcal{S}_0 \stackrel{\text{def}}{=} \{(\boldsymbol{p}, a) \in \mathcal{H}_0 \times \mathbb{P}^1 : \ a \notin \Psi(\boldsymbol{p}) \}$$

(a Zariski-open subset of $\mathcal{H} \times \mathbb{P}^1$) and let $\mathcal{T}_0$ be the inverse image of $\mathcal{S}_0$ in $\mathcal{T}$. Then G acts freely on $\mathcal{T}_0$ (because if a point of $\mathcal{T}_{\boldsymbol{p}}$, $\boldsymbol{p} \in \mathcal{H}_0$, is fixed by a non-trivial element of G then it is a branch point of $\varphi_{\boldsymbol{p}}$). The induced map $\mathcal{T}_0/G \to \mathcal{S}_0$ is bijective, hence homeomorphic (since a continuous proper bijection between locally compact Hausdorff spaces is homeomorphic). Thus the map $\mathcal{T}_0 \to \mathcal{S}_0$ is an (unramified) covering. Let $\mathcal{U}_0$ be the image of $\mathcal{S}_0$ in $\mathcal{U}(r+1)$, under the map $\Psi\times$ id. (Thus $\mathcal{U}_0$ is the set of all $(\boldsymbol{a}, a) \in \mathcal{U}(r+1)$ with $\boldsymbol{a} \in \Psi(\mathcal{H}_0)$, hence $\mathcal{U}_0$ is Zariski-open in $\mathcal{U}(r+1)$). Then the map $\Phi\colon \mathcal{T} \to \mathcal{U}_r \times \mathbb{P}^1$ restricts to a covering $\mathcal{T}_0 \to \mathcal{U}_0$.

2.6 Now the moduli space $\mathcal{T}_r(G)$ from §1 makes its appearance. Recall from 2.5 that if $t \in \mathcal{T}_0$, then for $\boldsymbol{p} = \pi\Lambda(t)$ the map $\varphi_{\boldsymbol{p}}\colon \mathcal{T}_{\boldsymbol{p}} \to \mathbb{P}^1$ is a Galois cover of $\mathbb{P}^1$ with r branch points, and $h_{\boldsymbol{p}}\colon \operatorname{Aut}(\mathcal{T}_{\boldsymbol{p}}/\mathbb{P}^1) \to G$ is an isomorphism. Further, t is a point of $\mathcal{T}_{\boldsymbol{p}}$ that does not lie over a branch point of $\varphi_{\boldsymbol{p}}$. Hence the triple $(\varphi_{\boldsymbol{p}}, h_{\boldsymbol{p}}, t)$ represents a point of $\mathcal{T}_r(G)$.

LEMMA: *The map* $\Omega\colon \mathcal{T}_0 \to \mathcal{T}_r(G)$, $t \mapsto |\varphi_{\boldsymbol{p}}, h_{\boldsymbol{p}}, t|$, *where* $\boldsymbol{p} = \pi\Lambda(t)$, *is continuous, even a local homeomorphism. Also the induced map* $\omega\colon \mathcal{H}_0 \to \mathcal{H}^{\text{in}}$, $\boldsymbol{p} \mapsto |\varphi_{\boldsymbol{p}}, h_{\boldsymbol{p}}|$, *is a local homeomorphism.*

Proof: Fix some $t \in \mathcal{T}_0$, let $(\boldsymbol{p}, a_0)$ be its image in $\mathcal{H}_0 \times \mathbb{P}^1$. Write $\Psi(\boldsymbol{p}) = \{a_1, ..., a_r\}$. Since the map $\mathcal{T} \to \mathcal{U}_r \times \mathbb{P}^1$ is a local homeomorphism on $\mathcal{T}_0$, there is a neighborhood $\mathcal{M}$ of t in $\mathcal{T}_0$ that is mapped homeomorphically to a neighborhood $\mathcal{U}$ of $(\{a_1, ..., a_r\}, a_0)$ in $\mathcal{U}_r \times \mathbb{P}^1$. We can take $\mathcal{U}$ of the following form: It consists of all $(\{\tilde{a}_1, ..., \tilde{a}_r\}, \tilde{a}_0)$ with $\tilde{a}_i \in D_i$ for $i = 0, ..., r$, where $D_0, ..., D_r$ are disjoint discs on $\mathbb{P}^1$ around $a_0, ..., a_r$. Then $\mathcal{N} = \pi\Lambda(\mathcal{M})$ (the image of $\mathcal{M}$ in $\mathcal{H}_0$) is mapped by Ψ homeomorphically to a subset $\cong D_1 \times \cdots \times D_r$ of $\mathcal{U}_r$. Thus $\mathcal{N}$ is contractible. Further, the set $\mathcal{N} \times \mathbb{P}^1 \setminus (D_1 \cup \cdots \cup D_r)$ is contained in $\mathcal{S}_0$; denote its inverse image in $\mathcal{T}_0$ by $\mathcal{N}'$.

For any $\tilde{t} \in \mathcal{M}$, let $(\tilde{\boldsymbol{p}}, \tilde{a}_0)$ be its image in $\mathcal{H}_0 \times \mathbb{P}^1$. Write $\Psi(\tilde{\boldsymbol{p}}) = \{\tilde{a}_1, ..., \tilde{a}_r\}$. The image $|\varphi_{\tilde{\boldsymbol{p}}}, h_{\tilde{\boldsymbol{p}}}, \tilde{t}|$ of $\tilde{t}$ in $\mathcal{T}_r(G)$ is associated to the triple $(\{\tilde{a}_1, ..., \tilde{a}_r\}, \tilde{a}_0, \tilde{f})$ (see 1.4), where

$$\tilde{f}\colon\ \pi_1(\mathbb{P}^1 \setminus \{\tilde{a}_1, ..., \tilde{a}_r\}, \tilde{a}_0) \to \operatorname{Aut}(\mathcal{T}_{\tilde{\boldsymbol{p}}}/\mathbb{P}^1) \cong G$$

is given as follows: The isomorphism with G comes from restriction of the action of G on $\mathcal{T}$, and the surjection $\pi_1(\mathbb{P}^1 \setminus \{\tilde{a}_1, ..., \tilde{a}_r\}, \tilde{a}_0) \to \operatorname{Aut}(\mathcal{T}_{\tilde{\boldsymbol{p}}}/\mathbb{P}^1)$ corresponds to the base point $\tilde{t} \in \mathcal{T}_{\tilde{\boldsymbol{p}}}$ over $\tilde{a}_0$ (see 1.4).

Since $\mathcal{N}$ and D_0 are contractible, we have a canonical isomorphism

$$(3) \qquad \pi_1(\mathcal{N} \times \mathbb{P}^1 \setminus (D_1 \cup \cdots \cup D_r),\ (\boldsymbol{p}, a_0)) \ \cong\ \pi_1(\mathbb{P}^1 \setminus \{\tilde{a}_1, ..., \tilde{a}_r\}, \tilde{a}_0)$$

Via this isomorphism, the natural action of the left fundamental group on the fiber in $\mathcal{N}'$ over $(\boldsymbol{p}, a_0)$ corresponds to the action of the right group on the fiber in $\mathcal{T}_{\tilde{\boldsymbol{p}}}$ over $\tilde{a}_0$. The latter action is transitive (since $\mathcal{T}_{\tilde{\boldsymbol{p}}}$ is connected), hence also the former action is transitive, and $\mathcal{N}'$ is connected. Identify the groups $\mathrm{Aut}(\mathcal{N}'/(\mathcal{N} \times \mathbb{P}^1 \setminus (D_1 \cup \ldots \cup D_r)))$ and $\mathrm{Aut}(\mathcal{T}_{\tilde{\boldsymbol{p}}}/\mathbb{P}^1)$ canonically with G (via restriction of the action of G on $\mathcal{T}$).

Via (3), the surjection $\tilde{f}\colon \pi_1(\mathbb{P}^1 \setminus \{\tilde{a}_1, \ldots, \tilde{a}_r\}, \tilde{a}_0) \to G$ that corresponds to the base point $\tilde{t} \in \mathcal{T}_{\tilde{\boldsymbol{p}}}$ is identified with the surjection $\pi_1(\mathcal{N} \times \mathbb{P}^1 \setminus (D_1 \cup \cdots \cup D_r), (\boldsymbol{p}, a_0)) \to G$ that corresponds to the base point $t \in \mathcal{N}'$. This means that the surjections $\tilde{f}$ (as $\tilde{t}$ ranges over $\mathcal{M}$) all correspond to each other under the isomorphisms (1). This proves that Ω maps $\mathcal{M}$ homeomorphically onto the basic neighborhood $\mathcal{M}(D_0, \ldots, D_r)$ of $\Omega(t)$ in $\mathcal{T}_r(G)$ (from 1.5). Similarly, ω maps $\mathcal{N}$ homeomorphically onto the basic neighborhood $\mathcal{N}(D_1, \ldots, D_r)$ of $\omega(\boldsymbol{p})$ in $\mathcal{H}^{\mathrm{in}}$. This completes the proof. ∎

2.7 Set $\mathcal{T}_0' = \Omega(\mathcal{T}_0)$, $\mathcal{H}_0' = \omega(\mathcal{H}_0)$. These are open subsets of $\mathcal{T}_r(G)$ and $\mathcal{H}^{\mathrm{in}}$, respectively (by the above Lemma). The covering $\mathcal{T}_0 \to \mathcal{U}_0$ (from 2.5) is the composition of $\Omega\colon \mathcal{T}_0 \to \mathcal{T}_0'$ with the covering $\Phi^{(r)}\colon \mathcal{T}_r(G) \to \mathcal{U}(r+1)$ (from 1.5); this is immediate from the definitions. Since Ω is a continuous map between (complex) manifolds, it follows that $\Omega\colon \mathcal{T}_0 \to \mathcal{T}_0'$ is a covering. Analogously, the covering $\Psi\colon \mathcal{H}_0 \to \Psi(\mathcal{H}_0)$ is the composition of $\omega\colon \mathcal{H}_0 \to \mathcal{H}^{\mathrm{in}}$ with the covering $\Psi^{\mathrm{in}}\colon \mathcal{H}^{\mathrm{in}} \to \mathcal{U}_r$; hence also $\omega\colon \mathcal{H}_0 \to \mathcal{H}_0'$ is a covering. For later use, we record the formulas

$$(4) \qquad \Phi|_{\mathcal{T}_0} = \Phi^{(r)} \circ \Omega, \qquad \Psi|_{\mathcal{H}_0} = \Psi^{\mathrm{in}} \circ \omega$$

It is now essentially clear how to complete the proof of the main theorem: Transfer the variety structure on $\mathcal{T}_0$ to $\mathcal{T}_0'$ via the covering Ω. By uniqueness of normalization, also the closure of $\mathcal{T}_0'$ in $\mathcal{T}_r(G)$ (a union of components of $\mathcal{T}_r(G)$) inherits this variety structure. Complete the proof by repeating this to cover all components of $\mathcal{T}_r(G)$.

However, there are still many details to check. Things work out much nicer if we assume that G has trivial center. This is anyway the case used in most applications (see [FV2], [Vö1], [Vö3]). The general case can be reduced to the case of trivial center as in [FV1, 5.2]: Choose a group $\hat{G}$ with trivial center and with $r-1$ generators that maps surjectively to G. Then use the natural map $\mathcal{T}_r(\hat{G}) \to \cup_{s=2}^r \mathcal{T}_s(G)$ to transfer the variety structure from $\mathcal{T}_r(\hat{G})$ to $\mathcal{T}_r(G)$; i.e., $\mathcal{T}_r(G)$ gets the structure as quotient variety of a closed and open subvariety of $\mathcal{T}_r(\hat{G})$. More precisely, let $\hat{C}$ be a $\mathbb{Q}$-irreducible component of $\mathcal{T}_r(\hat{G})$ that maps to $\mathcal{T}_r(G)$, and let C be the image of $\hat{C}$ in $\mathcal{T}_r(G)$. Then C is a union of connected components of $\mathcal{T}_r(G)$, and $\hat{C} \to C$ is a (unramified) covering, compatible with the coverings ($\Phi^{(r)}$ and its analogue) from $\hat{C}$ and C to $\mathcal{U}(r+1)$. Let K be the Galois closure of the field extension $\mathbb{Q}(\hat{C})/\mathbb{Q}(\mathcal{U}(r+1))$. Let G_0 be the corresponding Galois group, and let C_0 be the normalization of $\mathcal{U}(r+1)$ in K. The corresponding map $C_0 \to \mathcal{U}(r+1)$ is a covering, and G_0 acts canonically as a group of automorphisms of this covering such that the induced map $C_0/G_0 \to \mathcal{U}(r+1)$ is an isomorphism. Since the map $C_0 \to \mathcal{U}(r+1)$ factorizes as

$$C_0 \to \hat{C} \to C \to \mathcal{U}(r+1)$$

we have $C_0/G_1 \cong C$ for some subgroup G_1 of G_0. Now equip C with the structure as quotient variety by the fixed point free action of G_1. Repeat this procedure to cover all components of $\mathcal{T}_r(G)$.

So from now on we assume G has trivial center. Then we can assume (see 2.2): The field k (from 2.3) is the fixed field of the group of all $\beta \in \mathrm{Aut}(\mathbb{C})$ with

$$|\varphi_0, h|^\beta = |\varphi_0, h|$$

where h is any isomorphism $\mathrm{Aut}(X_0/\mathbb{P}^1) \to G$.

2.8 From the definitions it is easy to check that

$$\omega(\boldsymbol{p}^\beta) = \omega(\boldsymbol{p})^\beta, \quad \Omega(t^\beta) = \Omega(t)^\beta$$

for all $\boldsymbol{p} \in \mathcal{H}_0$, $t \in \mathcal{T}_0$, $\beta \in \mathrm{Aut}(\mathbb{C})$. Thereby, β acts on $\mathcal{H}^{\text{in}}$ as defined in 2.2, and acts on $\mathcal{T}_r(G)$ in the analogous way: Sending $|\varphi, h, p|$ to $|\varphi^\beta, h\beta^{-1}, p^\beta|$.

2.9 Consider again the generic point $\boldsymbol{p}_0$ of $\mathcal{H}$ from 2.4. It lies on the Zariski-open subset $\mathcal{H}_0$. For the generic cover φ_0: $X_0 \to \mathbb{P}^1$ (with which we started our development in 2.3), we have

$$\omega(\boldsymbol{p}_0) = |\varphi_0, h_0|$$

for some isomorphism h_0: $\mathrm{Aut}(X_0/\mathbb{P}^1) \to G$. For this it suffices to show that the cover $\varphi_{\boldsymbol{p}_0}$ is equivalent to φ_0. This follows from the fact that both covers correspond to the function field extension $L/k(x)$ from 2.3 (clear by construction).

2.10 Now assume that $\boldsymbol{p}_1$ is a point of $\mathcal{H}_0$ with $\omega(\boldsymbol{p}_1) = \omega(\boldsymbol{p}_0)$. Then $\Psi(\boldsymbol{p}_1) = \Psi(\boldsymbol{p}_0)$ $(= \mathbf{t})$ by (4). Since $\boldsymbol{p}_0$ is generic, all points in the fiber $\Psi^{-1}(\mathbf{t})$ are conjugate under $\mathbf{G}_{k_0}$, where $k_0 = \mathbb{Q}(\mathbf{t}) = \mathbb{Q}(x_1, ..., x_r)$ as in 2.3. Thus there is $\beta \in \mathrm{Aut}(\mathbb{C})$ sending $\boldsymbol{p}_0$ to $\boldsymbol{p}_1$. Then β also sends $\omega(\boldsymbol{p}_0)$ to $\omega(\boldsymbol{p}_1)$ (by 2.8). But we assumed $\omega(\boldsymbol{p}_1) = \omega(\boldsymbol{p}_0)$, hence β fixes $\omega(\boldsymbol{p}_0)$. Since $\omega(\boldsymbol{p}_0) = |\varphi_0, h_0|$ (by 2.9), it follows from 2.7 that β is the identity on k. But $k = \mathbb{Q}(\boldsymbol{p}_0)$, hence β fixes $\boldsymbol{p}_0$. Thus $\boldsymbol{p}_1 = \boldsymbol{p}_0$.

We have proved that if $\omega(\boldsymbol{p}_1) = \omega(\boldsymbol{p}_0)$ then $\boldsymbol{p}_1 = \boldsymbol{p}_0$ (for any $\boldsymbol{p}_1 \in \mathcal{H}_0$). Thus the covering $\omega: \mathcal{H}_0 \to \mathcal{H}'_0$ has degree 1, hence is an analytic isomorphism. Then also $\Omega: \mathcal{T}_0 \to \mathcal{T}'_0$ has degree 1, thus is an analytic isomorphism.

LEMMA: *The map Ω (resp., ω) is an analytic isomorphism from $\mathcal{T}_0$ (resp., $\mathcal{H}_0$) onto the open subset $\mathcal{T}'_0$ (resp., $\mathcal{H}'_0$) of $\mathcal{T}_r(G)$ (resp., $\mathcal{H}^{\mathrm{in}}$). The closure $\mathcal{T}'$ (resp., $\mathcal{H}'$) of $\mathcal{T}'_0$ (resp., $\mathcal{H}'_0$) is a union of connected components of $\mathcal{T}_r(G)$ (resp., $\mathcal{H}^{\mathrm{in}}$). And $\mathcal{T}' \setminus \mathcal{T}'_0$ (resp., $\mathcal{H}' \setminus \mathcal{H}'_0$) is a nowhere dense analytic subset of $\mathcal{T}'$ (resp., $\mathcal{H}'$).*

Proof: It remains to prove the last two assertions. We do the case of $\mathcal{T}'$. The other case is analogous.

Recall that $\mathcal{U}_0$ is Zariski-open in $\mathcal{U}(r+1)$. Thus the complement of $\mathcal{U}_0$ is an algebraic, hence analytic (proper) subset of $\mathcal{U}(r+1)$. Thus $\mathcal{U}_0$ is connected [GR, Ch. 7,§1.3].

The restriction of $\Phi^{(r)}$ yields a covering $\mathcal{T}'_0 \to \mathcal{U}_0$ (by (4), since $\Phi|_{\mathcal{T}_0}: \mathcal{T}_0 \to \mathcal{U}_0$ is a covering). Then each component C' of $\mathcal{T}'_0$ still covers the connected $\mathcal{U}_0$. Let C be the component of $\mathcal{T}_r(G)$ that contains C'. Since C' covers $\mathcal{U}_0$ and is open in C, it follows that C' is a component of the full inverse image C'' of $\mathcal{U}_0$ in C.

Since $\mathcal{U}(r+1) \setminus \mathcal{U}_0$ is an analytic subset of $\mathcal{U}(r+1)$, the set $C \setminus C''$ is a (proper) analytic subset of C. Since C is a connected complex manifold, C'' is connected [GR, Ch. 7,§1.3]. Hence $C' = C''$, which implies that C' is dense in C. Thus C is contained in $\mathcal{T}'$, and it follows that $\mathcal{T}'$ is the union those components C of $\mathcal{T}_r(G)$. Finally, $\mathcal{T}' \setminus \mathcal{T}'_0$ is the union of the $C \setminus C''$, hence is nowhere dense analytic. ∎

2.11 Set $\tilde{\mathcal{T}} = \Phi^{-1}(\mathcal{U}(r+1))$, the inverse image of $\mathcal{U}(r+1)$ $(\subset \mathcal{U}_r \times \mathbb{P}^1)$ in $\mathcal{T}$. This is an open subvariety of $\mathcal{T}$ defined over $\mathbb{Q}$, containing $\mathcal{T}_0 = \Phi^{-1}(\mathcal{U}_0)$.

LEMMA: *The analytic isomorphisms $\Omega: \mathcal{T}_0 \to \mathcal{T}'_0$ and $\omega: \mathcal{H}_0 \to \mathcal{H}'_0$ extend uniquely to analytic maps $\tilde{\Omega} : \tilde{\mathcal{T}} \to \mathcal{T}'$ and $\tilde{\omega}: \mathcal{H} \to \mathcal{H}'$, respectively. We have*

$\Phi|_{\tilde{\mathcal{T}}} = \Phi^{(r)} \circ \tilde{\Omega}$ *and* $\Psi = \Psi^{\text{in}} \circ \tilde{\omega}$ *and* $(\tilde{\omega} \times \text{id}) \circ \Lambda|_{\tilde{\mathcal{T}}} = \Lambda_r \circ \tilde{\Omega}$.

Proof: Again we prove this only for Ω, the other case is analogous.

Since $\mathcal{T}'$ is a union of components of $\mathcal{T}_r(G)$, the covering $\Phi^{(r)}: \mathcal{T}_r(G) \to \mathcal{U}(r+1)$ restricts to a covering $\Phi': \mathcal{T}' \to \mathcal{U}(r+1)$. Let $t \in \tilde{\mathcal{T}} \setminus \mathcal{T}_0$, and let V be an open ball around $\Phi(t)$ in $\mathcal{U}(r+1)$. Since $\Phi': \mathcal{T}' \to \mathcal{U}(r+1)$ is a covering, we may assume that

$$(\Phi')^{-1}(V) = \bigcup_{i=1}^{n} W_i$$

where the W_i are disjoint open sets in $\mathcal{T}'$, and Φ' maps each W_i homeomorphically onto V.

Let U be the connected component of $\Phi^{-1}(V)$ that contains t. The space $\mathcal{T}$ is a normal algebraic variety (by definition), hence the underlying complex space is also normal [SGA1, Exp. XII, Prop. 2.1]. Then also its open subset $\Phi^{-1}(V)$ is a normal complex space. The components of a normal complex space are open [GR, p. 171], hence U is open in $\mathcal{T}$. Then U is also a normal complex space.

Since $\mathcal{U}(r+1) \setminus \mathcal{U}_0$ is a nowhere dense analytic subset of $\mathcal{U}(r+1)$ (see 2.10), its inverse image $\tilde{\mathcal{T}} \setminus \mathcal{T}_0$ under Φ is a nowhere dense analytic subset of $\tilde{\mathcal{T}}$. Hence with U also $U \cap \mathcal{T}_0$ is connected [GR, p.145]. Since Ω maps $U \cap \mathcal{T}_0$ into $(\Phi')^{-1}(V)$ (by (4)), it follows that Ω maps $U \cap \mathcal{T}_0$ into some W_i. Thus for all $s \in U \cap \mathcal{T}_0$ we have

$$\Omega(s) = (\Phi'|_{W_i})^{-1} \circ \Phi(s).$$

Now we can extend Ω to U by the same formula (because the right hand side is defined for all $s \in U$). This extension is unique because $U \cap \mathcal{T}_0$ is dense in U. Thus all these extensions glue to give the desired $\tilde{\Omega}$.

The extension $\tilde{\omega}$ is obtained analogously. The other assertions in the Lemma follow by continuity (since the respective formulas hold for Ω and ω, cf. (4) and 2.6). ∎

2.12 Lemma: *The analytic isomorphisms* $\Omega: \mathcal{T}_0 \to \mathcal{T}_0'$ *and* $\omega: \mathcal{H}_0 \to \mathcal{H}_0'$ *extend uniquely to analytic isomorphisms* $\tilde{\Omega}: \tilde{\mathcal{T}} \to \mathcal{T}'$ *and* $\tilde{\omega}: \mathcal{H} \to \mathcal{H}'$, *respectively.*

Proof: Again we consider only the case of Ω.

In the set-up of 2.4, let $\bar{\mathcal{T}}$ be the normalization of $\mathbb{P}^r \times \mathbb{P}^1$ in the extension field L of $\mathbb{Q}(x_1, ..., x_r, x)$, and let

$$\bar{\Phi}: \bar{\mathcal{T}} \to \mathbb{P}^r \times \mathbb{P}^1$$

be the corresponding map. Then $\bar{\Phi}^{-1}(\mathcal{U}_r \times \mathbb{P}^1)$ is normal and maps through a finite surjective morphism to $\mathcal{U}_r \times \mathbb{P}^1$, hence it is isomorphic to the normalization of $\mathcal{U}_r \times \mathbb{P}^1$ in L [Mu, p. 277]). Thus we may assume $\mathcal{T} = \bar{\Phi}^{-1}(\mathcal{U}_r \times \mathbb{P}^1)$, $\Phi = \bar{\Phi}|_{\mathcal{T}}$. Note that $\bar{\mathcal{T}}$ is a projective variety [Mu, p. 280]. Further, $\bar{\Phi}$ is a finite, hence affine map [Mu, p. 172, p. 277]; i.e., for any Zariski-open, affine subset $\mathcal{A}$ of $\mathbb{P}^r \times \mathbb{P}^1$, the inverse image $\bar{\Phi}^{-1}(\mathcal{A})$ is again affine.

Recall from 2.10 that $\mathcal{T}_0' = \mathcal{T}' \setminus A$, where A is a nowhere dense analytic subset of the manifold $\mathcal{T}'$. Fix a point $a \in A$, and let $\mathcal{A}$ be a Zariski-open, affine neighborhood of $\Phi'(a)$ in $\mathbb{P}^r \times \mathbb{P}^1$. Let D^0 (resp., D') be an open (resp., compact) ball in the manifold $\mathcal{T}'$ around a, with $D^0 \subset D' \subset (\Phi')^{-1}(\mathcal{A})$. Then $\Phi'(D')$ is a compact subset of $\mathcal{A}$ ($\subset \mathbb{P}^r \times \mathbb{P}^1$), hence $D \overset{\text{def}}{=} \bar{\Phi}^{-1}(\Phi'(D'))$ is a closed subset of $\bar{\mathcal{T}}$ (in the complex topology). Hence D is compact (since $\bar{\mathcal{T}}$ is a projective variety).

Now embed the affine variety $\bar{\Phi}^{-1}(\mathcal{A})$ into some affine space $\mathbb{C}^n$. The inclusions $\Omega^{-1}(D^0 \cap \mathcal{T}_0') \subset D \subset \bar{\Phi}^{-1}(\mathcal{A})$ imply $\Omega^{-1}|_{D^0 \cap \mathcal{T}_0'} = (\omega_1, \ldots, \omega_n)$ for holomorphic (complex-valued) functions ω_i on $D^0 \cap \mathcal{T}_0'$. These functions ω_i are bounded because D is compact. By Riemann's extension theorem [GR, p. 131], it follows that the ω_i extend to holomorphic functions on D^0. (Recall that $D^0 \cap \mathcal{T}_0' = D^0 \setminus A$, where A is a nowhere dense analytic subset of $\mathcal{T}'$).

Then Ω^{-1} extends to an analytic map on D^0 with values in $\bar{\mathcal{T}}$. Because of the uniqueness of such an extension, these local extensions glue together to give a global extension of Ω^{-1} to an analytic map $\Omega'\colon \mathcal{T}' \to \bar{\mathcal{T}}$. By continuity, we must have $\Omega' \circ \tilde{\Omega} = \mathrm{id}_{\tilde{\mathcal{T}}}$, and $\bar{\Phi} \circ \Omega' = \Phi'$. The latter implies that the image of Ω' lies in $\tilde{\mathcal{T}}$ ($= \bar{\Phi}^{-1}(\mathcal{U}(r+1))$). Then also $\tilde{\Omega} \circ \Omega' = \mathrm{id}_{\mathcal{T}'}$, hence $\tilde{\Omega}$ is an isomorphism from $\tilde{\mathcal{T}}$ onto $\mathcal{T}'$. ∎

It follows from Lemma 2.11 and 2.12 that we can actually take $\mathcal{H}_0 = \mathcal{H}$ and $\mathcal{T}_0 = \tilde{\mathcal{T}}$ in the above; i.e., the maps ω and Ω can be defined on all of $\mathcal{H}$ and $\tilde{\mathcal{T}}$, respectively, by the formulas from Lemma 2.6. (Namely, the spaces $\mathcal{T}'$ and $\mathcal{H}'$ have the properties required for $\mathcal{T}_0$ and $\mathcal{H}_0$, and via the isomorphisms $\tilde{\Omega}$ and $\tilde{\omega}$ these properties are inherited by $\tilde{\mathcal{T}}$ and $\mathcal{H}$). Of course, the so-defined maps $\omega\colon \mathcal{H} \to \mathcal{H}'$ and $\Omega\colon \tilde{\mathcal{T}} \to \mathcal{T}'$ are analytic isomorphisms. These isomorphisms equip those components of our moduli spaces $\mathcal{H}^{\mathrm{in}}$ and $\mathcal{T}_r(G)$ that are contained in $\mathcal{H}'$ and $\mathcal{T}'$, respectively, with the desired structure of variety defined over $\mathbb{Q}$ (see below). First we show that all components of the moduli spaces are covered by this procedure.

2.13 The space $\mathcal{H}'$ contains the point $\omega(\boldsymbol{p}_0) = |\varphi_0, h_0|$ (from 2.9), where φ_0 is the original cover that was the starting point of our construction in 2.3. Applying the automorphisms δ_A of $\mathcal{H}^{\text{in}}$ from 1.5 (that commute with the Aut($\mathbb{C}$)-action on $\mathcal{H}^{\text{in}}$), we transfer the above variety structure from $\mathcal{H}'$ to the $\delta_A(\mathcal{H}')$. The union of these $\delta_A(\mathcal{H}')$ contains all points $|\varphi_0, h|$ (with h any isomorphism $\text{Aut}(\varphi_0) \to G$). This implies the following: As φ_0 ranges over all Galois covers of $\mathbb{P}^1$ with group G that are ramified exactly at the points $t_1, ..., t_r$, the corresponding spaces $\mathcal{H}'$ together with their δ_A-conjugates contain all points of $\mathcal{H}^{\text{in}}$ that lie over the (base) point $\mathbf{t}$. Hence the union of these spaces is all of $\mathcal{H}^{\text{in}}$ (since they are unions of components of $\mathcal{H}^{\text{in}}$). This equips $\mathcal{H}^{\text{in}}$ with the desired variety structure. The procedure for $\mathcal{T}_r(G)$ is analogous (since over each component of $\mathcal{H}^{\text{in}}$ there is a unique component of $\mathcal{T}_r(G)$).

Now we have equipped $\mathcal{H}^{\text{in}}$ and $\mathcal{T}_r(G)$ with a structure as variety defined over $\mathbb{Q}$. By 2.8, the formulas from the main theorem for the action of $\beta \in \text{Aut}(\mathbb{C})$ hold. The $\mathbb{Q}$-variety structure on our spaces is uniquely determined by these formulas, and by the $\mathbb{C}$-variety structure. Uniqueness of the $\mathbb{C}$-variety structure follows from the following

LEMMA: *Let Y be an open subvariety of $\mathbb{P}^r$ or $\mathbb{P}^r \times \mathbb{P}^1$, and let Y_i ($i = 1, 2$) be the normalization of Y in the finite field extension L_i of $\mathbb{C}(Y)$. Let $f_i\colon Y_i \to Y$ be the corresponding map, and assume f_i is an unramified covering in the complex topology. If $\epsilon\colon Y_1 \to Y_2$ is an analytic isomorphism with $f_2 \circ \epsilon = f_1$, then ϵ is an algebraic isomorphism (defined over $\mathbb{C}$).*

Proof: This follows from the uniqueness part of the generalized Riemann existence theorem (cf. [S, Th. 6.1.4]). The uniqueness is much less deep than the existence part. For completeness, we sketch a direct proof in our special case.

STEP 1: First note that the problem is local on Y (see [Mu, p. 278]), so we may assume Y is affine. Then also Y_i is affine, and its coordinate ring $\mathbb{C}[Y_i]$ is the integral closure of $\mathbb{C}[Y]$ in L_i. Once we have shown that the map $h \mapsto h\epsilon^{-1}$ maps $\mathbb{C}[Y_1]$ into $\mathbb{C}[Y_2]$, it follows that this map is a ring isomorphism, and ϵ equals the corresponding algebraic map $Y_1 = \text{Spec}(\mathbb{C}[Y_1]) \to Y_2 = \text{Spec}(\mathbb{C}[Y_2])$.

STEP 2: Use Riemann's extension theorem (as in 2.12) to extend ϵ to an analytic isomorphism $\bar{Y}_1 \to \bar{Y}_2$, where $\bar{Y}_i$ is the normalization of $\bar{Y} \overset{\text{def}}{=} \mathbb{P}^r$ or $\overset{\text{def}}{=} \mathbb{P}^r \times \mathbb{P}^1$, respectively, in L_i. It is well-known that all meromorphic functions on $\bar{Y}$ are rational [GR, p. 186]. To deduce the corresponding statement for $\bar{Y}_i$, note that the field of meromorphic functions on $\bar{Y}_i$ has degree $\leq n$ over the field of meromorphic functions on $\bar{Y}$, where n is the degree of the analytic (ramified) covering $\bar{Y}_i \to \bar{Y}$.

It follows that the map $h \mapsto h\epsilon^{-1}$ maps $\mathbb{C}(\bar{Y}_1) = \mathbb{C}(Y_1) = L_1$ into $\mathbb{C}(\bar{Y}_2) = \mathbb{C}(Y_2) = L_2$. Then it also maps $\mathbb{C}[Y_1]$ into $\mathbb{C}[Y_2]$ (because $\mathbb{C}[Y_i]$ is the integral closure of $\mathbb{C}[Y]$ in L_i). Now Step 1 applies, and we are done. ∎

The Lemma implies that the maps ϵ_A and δ_A are algebraic morphisms (defined over $\mathbb{C}$). They are actually defined over $\mathbb{Q}$: This follows from the formulas in the main theorem for the action of $\beta \in \mathrm{Aut}(\mathbb{C})$ (as in [FV1, 6.2]). This completes the proof of the main theorem.

3. Coverings of affine space minus the discriminant locus

In this section we get to the corollaries about coverings of $\mathcal{O}_r = \mathbf{A}^r \setminus D_r$. Recall that we view $\mathcal{O}_r$ as the space of all subsets of $\mathbb{C}$ of cardinality r.

3A. Coverings determined by braid group action on generating systems.

3.1 Consider the space

$$\mathcal{H}_r(G) \stackrel{\text{def}}{=} \Lambda_r^{-1}(\mathcal{H}^{\text{in}} \times \{\infty\}),$$

the subspace of $\mathcal{T}_r(G)$ consisting of all $|\varphi, h, p|$ where $\varphi\colon X \to \mathbb{P}^1$ is a (Galois) cover of $\mathbb{P}^1$ with r branch points, none of which equals ∞, $h\colon \mathrm{Aut}(X/\mathbb{P}^1) \to G$ is an isomorphism, and p is a point of X with $\varphi(p) = \infty$. As in 1.4, the points of $\mathcal{H}_r(G)$ can equally well be thought of as pairs $(\boldsymbol{a}, f)$, where $\boldsymbol{a} = \{a_1, \ldots, a_r\} \in \mathcal{O}_r$ and $f\colon \Gamma = \pi_1(\mathbb{P}^1 \setminus \boldsymbol{a}, \infty) \to G$ is a surjection that does not factor through the canonical map $\Gamma \to \Gamma_i$, for any i (for the definition of Γ_i, see 1.2).

The topology on $\mathcal{H}_r(G)$ induced from that of $\mathcal{T}_r(G)$ (see 1.5) has a basis consisting of the following sets $\mathcal{M}_\infty$: Choose pairwise disjoint open discs $D_1, \ldots, D_r$ in $\mathbb{C}$. Then $\mathcal{M}_\infty$ corresponds to the set of pairs $(\boldsymbol{a}, f)$ as above, where $|\boldsymbol{a} \cap D_i| = 1$ for all i, and where the maps f are all induced from a fixed surjection $\pi_1(\mathbb{P}^1 \setminus (D_1 \cup \cdots \cup D_r), \infty) \to G$, via the canonical isomorphism

$$\pi_1(\mathbb{P}^1 \setminus (D_1 \cup \cdots \cup D_r), \infty) \cong \pi_1(\mathbb{P}^1 \setminus \boldsymbol{a}, \infty)$$

One checks that the map $\Psi_\infty\colon \mathcal{H}_r(G) \to \mathcal{O}_r$ that sends $|\varphi, h, p|$ to the set $\boldsymbol{a}$ of branch points of φ, is a covering. Hence $\mathcal{H}_r(G)$ is a (closed) submanifold of $\mathcal{T}_r(G)$.

Denote the restriction to $\mathcal{H}_r(G)$ of the maps ϵ_A ($A \in \mathrm{Aut}(G)$) from 1.5 again by ϵ_A. These maps are automorphisms of the covering $\Psi_\infty \colon \mathcal{H}_r(G) \to \mathcal{O}_r$.

3.2 It follows from the main theorem that $\mathcal{H}_r(G)$ has a unique structure as algebraic variety defined over $\mathbb{Q}$ such that Ψ_∞ and all ϵ_A are algebraic morphisms defined over $\mathbb{Q}$, and the automorphisms of $\mathbb{C}$ act on the points $|\varphi, h, p|$ of $\mathcal{H}_r(G)$ as given in the main theorem.

3.3 The next goal is to determine the space $\mathcal{H}_r(G)$ topologically (as a covering of $\mathcal{O}_r$). Fix a base point $\boldsymbol{b} = \{b_1, ..., b_r\}$ in $\mathcal{O}_r$. The fundamental group

$$\mathcal{B}_r = \pi_1(\mathcal{O}_r, \boldsymbol{b})$$

of $\mathcal{O}_r$, based at $\boldsymbol{b}$, has generators $Q_1, ..., Q_{r-1}$ defined in [FV1, 1.3]. These generators satisfy the defining relations for the *Artin braid group* (and no others), hence $\mathcal{B}_r$ is isomorphic to the Artin braid group. This is all well-known (see [Ar]), but we will actually not need it here.

By covering space theory, each covering of $\mathcal{O}_r$ is determined (up to equivalence) by the natural permutation representation of the fundamental group $\pi_1(\mathcal{O}_r, \boldsymbol{b})$ on the fiber over $\boldsymbol{b}$. Recall that this permutation representation is given as follows: Each element of $\pi_1(\mathcal{O}_r, \boldsymbol{b})$, represented by a closed path c, maps a point $\boldsymbol{p}$ lying over $\boldsymbol{b}$ to the endpoint of the unique lift of c with initial point $\boldsymbol{p}$.

We will determine the permutation representation of $\mathcal{B}_r$ corresponding to the covering $\Psi_\infty : \mathcal{H}_r(G) \to \mathcal{O}_r$. First we give another description of the fiber $\Psi_\infty^{-1}(\boldsymbol{b})$.

3.4 Choose generators $\gamma_1, ..., \gamma_r$ of

$$\Gamma_0 = \pi_1(\mathbb{P}^1 \setminus \boldsymbol{b}, \infty)$$

as usual (i.e., γ_i is a loop going clockwise once around b_i, see e.g. [FV1, 1.3]). Then Γ_0 is a free group on the generators $\gamma_1, ..., \gamma_{r-1}$, and we have $\gamma_1 ... \gamma_r = 1$ (for suitable labelling of the b_i's).

Let $\mathcal{E}_r$ be the set of all $(g_1, ..., g_r) \in G^r$ with the following properties: $g_1 ... g_r = 1$, the group G is generated by $g_1, ..., g_r$, and $g_i \neq 1$ for all i. Then the surjections $f : \Gamma_0 \to G$ with $f(\gamma_i) \neq 1$ for all i correspond bijectively to the elements of $\mathcal{E}_r$ (where the tuple $(f(\gamma_1), ..., f(\gamma_r))$ is associated to f). These surjections f also correspond bijectively to the points in the fiber $\Psi_\infty^{-1}(\boldsymbol{b})$ (by 3.1); namely, the condition that f does not factor through the canonical map $\Gamma \to \Gamma_i$ is equivalent to $f(\gamma_i) \neq 1$ (see [FV1, 1.3]).

Thus we have set up a bijection between the set $\mathcal{E}_r$ and the fiber $\Psi_\infty^{-1}(\boldsymbol{b})$. Via this bijection, the covering $\Psi_\infty : \mathcal{H}_r(G) \to \mathcal{O}_r$ yields a permutation representation of $\mathcal{B}_r =< Q_1, ..., Q_{r-1} >$ on $\mathcal{E}_r$. The action of the Q_i's on $\mathcal{E}_r$ can be described explicitly:

THEOREM: *The covering $\Psi_\infty : \mathcal{H}_r(G) \to \mathcal{O}_r$ corresponds to the permutation representation of $\mathcal{B}_r =< Q_1, ..., Q_{r-1} >$ on $\mathcal{E}_r$ where Q_i $(i = 1, ..., r-1)$ sends $(g_1, ..., g_r) \in \mathcal{E}_r$ to:*

$$(g_1, ..., g_{i-1}, g_{i+1}, g_{i+1}^{-1} g_i g_{i+1}, ..., g_r)$$

The proof is the same as for the analogous result about the permutation representation of $\pi_1(\mathcal{U}_r, \boldsymbol{b})$ corresponding to the covering $\Psi^{\text{in}} : \mathcal{H}_r^{\text{in}}(G) \to \mathcal{U}_r$ (see [FV1, 1.3, 1.4]). To remind the reader, the group $\pi_1(\mathcal{U}_r, \boldsymbol{b})$ is the Hurwitz braid group, a quotient of the Artin braid group (see [BF]), and its permutation representation corresponding to Ψ^{in} is just the action on $\mathcal{E}_r/\ \text{Inn}(G)$ (the set of $\text{Inn}(G)$-orbits on $\mathcal{E}_r$) induced from the action of $\mathcal{B}_r$ on $\mathcal{E}_r$. Thereby, $\text{Inn}(G)$ is the group of inner automorphisms of G.

Actually, the new result implies that previous one, because the covering Ψ_∞ factorizes through Ψ^{in}.

3B. COMPONENTS OF THE COVERINGS, AND CONJUGACY CLASSES OF G.

We conclude the paper with a study of certain subspaces of $\mathcal{H}_r(G)$ that are associated to r-tuples of conjugacy classes of G.

3.5 Let $\mathcal{C} = (C_1, ..., C_r)$ be an r-tuple of conjugacy classes of G. The Nielsen class $\text{Ni}(\mathcal{C})$ is defined to be the set of all $(g_1, ..., g_r) \in \mathcal{E}_r$ for which there is a permutation $\pi \in S_r$ with $g_{\pi(i)} \in C_i$ for all i. Clearly, the set $\text{Ni}(\mathcal{C})$ is invariant under the above action of $\mathcal{B}_r$ on $\mathcal{E}_r$.

By covering space theory, the connected components of $\mathcal{H}_r(G)$ are in 1-1 correspondence with the orbits of $\mathcal{B}_r$ on $\mathcal{E}_r$: Each component intersects the fiber $\Psi_\infty^{-1}(\boldsymbol{b})$ (identified with $\mathcal{E}_r$) in the associated $\mathcal{B}_r$-orbit. Let $\mathcal{H}(\mathcal{C})$ be the union of the components of $\mathcal{H}_r(G)$ that correspond to the $\mathcal{B}_r$-orbits contained in $\text{Ni}(\mathcal{C})$.

On the fiber $\Psi_\infty^{-1}(\boldsymbol{b})$ (identified with $\mathcal{E}_r$) the map ϵ_A acts as follows: It sends $(g_1, ..., g_r)$ to $(A(g_1), ..., A(g_r))$. This is immediate from the definitions. It follows that ϵ_A leaves $\mathcal{H}(\mathcal{C})$ invariant if and only if the automorphism A of G permutes the conjugacy classes $C_1, ..., C_r$.

The definition of the spaces $\mathcal{H}(\mathcal{C})$ seems to depend on the choice of the base point $\boldsymbol{b}$. We show that this is not so, by giving another characterization that does not depend on the base point. First we need some topological preparations.

3.6 THE DISTINGUISHED GENERATOR. For $0 \leq s < t$, and $a \in \mathbb{C}$, consider the annulus

$$R = \{z \in \mathbb{C} : \ s < |z-a| < t\}.$$

If $\chi : R' \to R$ is a connected covering of finite degree e, define the *distinguished generator* α of the (cyclic) group $\mathrm{Aut}(R'/R)$ as follows: We know that the covering χ is equivalent to the covering $\chi_e : R \to R$, $z \mapsto (z-a)^e + a$. Thus there is a homeomorphism $\delta : R' \to R$ with $\chi_e \delta = \chi$. This δ induces an isomorphism between $\mathrm{Aut}(R'/R)$ and $\mathrm{Aut}(\chi_e : R \to R)$. Let $\alpha \in \mathrm{Aut}(R'/R)$ be the image of the following generator of $\mathrm{Aut}(\chi_e : R \to R)$: rotation in counter-clockwise direction around a by the angle $2\pi/e$.

Another choice of δ, say $\delta_1 : R' \to R$, results in an element α_1 that is conjugate to α under $\delta^{-1}\delta_1 \in \mathrm{Aut}(R'/R)$. Since $\mathrm{Aut}(R'/R)$ is abelian, we see that α does not depend on the choice of δ. Thus the distinguished generator is well-defined, and then also functorial: If the covering $R' \to R$ factors as $R' \to R'' \to R$, then the distinguished generator of $\mathrm{Aut}(R'/R)$ induces that of $\mathrm{Aut}(R''/R)$.

Let γ be a closed path in R, based at the point ρ, and going once in clockwise direction around a circle centered at a. Consider the homomorphism $\iota : \pi_1(R, \rho) \to \mathrm{Aut}(R'/R)$ associated to any point in $\chi^{-1}(\rho)$ (by 1.2). (Actually, each such point yields the same homomorphism, since $\mathrm{Aut}(R'/R)$ is abelian). Clearly, ι maps γ to the distinguished generator.

3.7 THE DISTINGUISHED INERTIAL GROUP GENERATORS. Let $\varphi : X \to \mathbb{P}^1$ be a Galois cover with branch points $a_1, ..., a_r$, and set $H \stackrel{\text{def}}{=} \mathrm{Aut}(X/\mathbb{P}^1)$. Let $D_1, ..., D_r$ be disjoint discs around $a_1, ..., a_r$, respectively. It is well-known that each connected component X_{ij} of $\varphi^{-1}(D_i)$ contains exactly one point p_{ij} over a_i. Hence the stabilizer of p_{ij} in H (called the inertial group of p_{ij}) fixes X_{ij}, and induces the full automorphism group of the unramified covering $X_{ij} \setminus \{p_{ij}\} \to D_i \setminus \{a_i\}$. The distinguished generator of this automorphism group (from 3.6) yields a *distinguished generator* α_{ij} *of the inertial group of* p_{ij}. If $\epsilon \in H$ maps p_{ij} to $p_{ij'}$ then ϵ conjugates α_{ij} into $\alpha_{ij'}$ (by the functoriality in 3.6). Since H

acts transitively on the points over a_i, it follows that the α_{ij} (for fixed i) form a single conjugacy class of H.

Clearly, the α_{ij} do not depend on the choice of $D_1, ..., D_r$.

3.8 Now we can give the base point free characterization of the subspace $\mathcal{H}(\mathcal{C})$ of $\mathcal{H}_r(G)$. Recall that $\mathcal{H}_r(G)$ is the set of all $|\varphi, h, p|$ where $\varphi : X \to \mathbb{P}^1$ is a (Galois) cover with r branch points, none of which equals ∞, $h : \mathrm{Aut}(X/\mathbb{P}^1) \to G$ is an isomorphism, and p is a point of X with $\varphi(p) = \infty$.

PROPOSITION: *Let $\mathcal{C} = (C_1, ..., C_r)$ be an r-tuple of conjugacy classes of G. Then $\mathcal{H}(\mathcal{C})$ is the subspace of $\mathcal{H}_r(G)$ consisting of all $|\varphi, h, p| \in \mathcal{H}_r(G)$ with the following property: The branch points of φ can be labelled as $a_1, ..., a_r$ such that the conjugacy class of distinguished inertial group generators over a_i corresponds to C_i under h (for $i = 1, ..., r$).*

Proof: Let $\mathcal{H}'(\mathcal{C})$ be the set of all $|\varphi, h, p| \in \mathcal{H}_r(G)$ with the above property.

CLAIM 1: *$\mathcal{H}'(\mathcal{C})$ is open and closed in $\mathcal{H}_r(G)$.* Since $\mathcal{H}_r(G)$ is the disjoint union of the $\mathcal{H}'(\mathcal{C})$ (with repetitions deleted), it suffices to show that $\mathcal{H}'(\mathcal{C})$ is open.

Fix a point $\boldsymbol{p} = |\varphi, h, p|$ of $\mathcal{H}'(\mathcal{C})$, and label the branch points of φ as $a_1, ..., a_r$, as in the Proposition. Choose disjoint discs $D_1, ..., D_r$ around $a_1, ..., a_r$, where ∞ lies in none of the D_i. Further, pick smaller discs $D_i' \subset D_i$ around a_i. Set $X_0 = \varphi^{-1}(\mathbb{P}^1 \setminus (D_1' \cup ... \cup D_r'))$.

By the definition of the topology on $\mathcal{H}_r(G)$, there is a neighborhood $\mathcal{N}$ of $\boldsymbol{p}$ such that each $\boldsymbol{p}' \in \mathcal{N}$ can be written in the form $\boldsymbol{p}' = |\varphi', h', p|$, where φ' has exactly one branch point in each D_i', and $(\varphi')^{-1}(\mathbb{P}^1 \setminus (D_1' \cup ... \cup D_r'))$ is identified with X_0. Further, h' corresponds to h under the isomorphisms

$$\mathrm{Aut}(X'/\mathbb{P}^1) \cong \mathrm{Aut}(X_0/\mathbb{P}^1) \cong \mathrm{Aut}(X/\mathbb{P}^1) \tag{5}$$

(that come from restriction to X_0).

Clearly, in the definition of the distinguished inertial group generators for φ, one can replace the punctured discs $D_i \setminus \{a_i\}$ by the annuli $D_i \setminus D_i'$. Since φ and φ' coincide over $D_i \setminus D_i'$, it follows that the distinguished inertial group generators for φ and φ' correspond via the isomorphisms (5). Thus all $\boldsymbol{p}' \in \mathcal{N}$ lie in $\mathcal{H}'(\mathcal{C})$. This proves Claim 1.

CLAIM 2: *$\mathcal{H}'(\mathcal{C})$ intersects the fiber $\Psi_\infty^{-1}(\boldsymbol{b})$ (identified with $\mathcal{E}_r$ as in 3.4) in the set $Ni(\mathcal{C})$.* Recall the identification of the fiber $\Psi_\infty^{-1}(\boldsymbol{b})$ with $\mathcal{E}_r$ from 3.4: Firstly, for $\boldsymbol{p} = |\varphi, h, p| \in \Psi_\infty^{-1}(\boldsymbol{b})$ consider the map $f : \Gamma_0 = \pi_1(\mathbb{P}^1 \setminus \boldsymbol{b}, \infty) \to G$, which is the composition of $\iota : \Gamma_0 \to \mathrm{Aut}(X/\mathbb{P}^1)$ (associated to the point $p \in \varphi^{-1}(\infty)$) with h. Then $\boldsymbol{p}$ is identified with the r-tuple $(g_1, ..., g_r) \in \mathcal{E}_r$, where $g_i = f(\gamma_i) = h\iota(\gamma_i)$.

Choose again disjoint discs $D_1, ..., D_r$ around $b_1, ..., b_r$. Fix some $i = 1, ..., r$. We may assume that γ_i goes on a path ω to a point inside D_i, then goes on a circular path γ once around b_i (in clockwise direction), and returns to ∞ via the inverse of ω. Let p' be the endpoint of the (unique) lift of ω to $X \setminus \varphi^{-1}(\boldsymbol{b})$ with initial point p. Then p' lies in a unique component C of $\varphi^{-1}(D_i)$. Let p'' be the unique point in $C \cap \varphi^{-1}(b_i)$. From 3.6, applied with $R = D_i \setminus \{b_i\}$, $R' = C \setminus \{p''\}$, it follows that $\iota(\gamma_i)$ is the distinguished generator for the inertial group of p''.

Thus the point $\boldsymbol{p} = |\varphi, h, p| \in \Psi_\infty^{-1}(\boldsymbol{b})$ lies in $\mathcal{H}'(\mathcal{C})$ if and only if $(g_1, ..., g_r) \in \mathrm{Ni}(\mathcal{C})$, because $g_i = h\iota(\gamma_i)$. If $\boldsymbol{p}$ is identified with $(g_1, ..., g_r)$ as above, Claim 2 follows.

Now we can conclude the proof: By Claim 1, $\mathcal{H}'(\mathcal{C})$ is a union of connected components of $\mathcal{H}_r(G)$. The same is true for $\mathcal{H}(\mathcal{C})$ (see 3.5). Furthermore, $\mathcal{H}'(\mathcal{C})$ and $\mathcal{H}(\mathcal{C})$ have the same intersection with the fiber $\Psi_\infty^{-1}(\boldsymbol{b})$ (by Claim 2 and the definition of $\mathcal{H}(\mathcal{C})$). This implies $\mathcal{H}(\mathcal{C}) = \mathcal{H}'(\mathcal{C})$. ∎

Remark: The r-tuple $(g_1, ..., g_r)$ (of distinguished inertial group generators) is usually called a *description of branch cycles* of the cover φ. The property in the above Proposition was expressed in [Fr] by saying that φ lies in the *Nielsen class of* $\mathcal{C}$. ∎

3.9 THE FIELD OF DEFINITION OF $\mathcal{H}(\mathcal{C})$. The absolutely irreducible components of $\mathcal{H}_r(G)$ (viewed as variety) coincide with the connected components (in the complex topology). Thus $\mathcal{H}(\mathcal{C})$ is a (closed and open) subvariety of $\mathcal{H}_r(G)$ defined over $\bar{\mathbb{Q}}$. We are going to determine how $\mathbf{G}_{\mathbb{Q}}$ permutes the $\mathcal{H}(\mathcal{C})$.

For each integer m let C_i^m be the conjugacy class of g^m with $g \in C_i$. The r-tuple $\mathcal{C} = (C_1, ..., C_r)$ is called *rational* if for each integer m prime to the order of G, we have that $C_1^m, ..., C_r^m$ is a permutation of $C_1, ..., C_r$. Set $\mathcal{C}^m = (C_1^m, ..., C_r^m)$.

THEOREM: *Let $\beta \in \mathbf{G}_{\mathbb{Q}}$, and let m be an integer such that β^{-1} acts on the $|G|$-th roots of unity as $\eta \mapsto \eta^m$. Then β maps $\mathcal{H}(\mathcal{C})$ to $\mathcal{H}(\mathcal{C}^m)$. Thus $\mathcal{H}(\mathcal{C})$ is* *defined over* $\mathbb{Q}_{ab}$ *(the field generated by all roots of unity). If* $\mathcal{C}$ *is rational then* $\mathcal{H}(\mathcal{C})$ *is even defined over* $\mathbb{Q}$.

Proof: Extend β to an automorphism of $\mathbb{C}$ (that we again denote by β). Consider any $\boldsymbol{p} = |\varphi, h, p| \in \mathcal{H}_r(G)$, and label the branch points $a_1, ..., a_r$ of $\varphi : X \to \mathbb{P}^1$ as in the above Proposition. By 3.2 we have $\boldsymbol{p}^\beta = |\varphi^\beta, h\beta^{-1}, p^\beta|$. Thus it suffices to show (by the Proposition) that for the cover $\varphi^\beta : X^\beta \to \mathbb{P}^1$, the class of distinguished inertial group generators over a_i^β corresponds to C_i^m under $h\beta^{-1}$. This follows from the *branch cycle argument* of [Fr, p.63]. For completeness, we give the argument in the following Lemma. ∎

LEMMA (The branch cycle argument): *Let* $a_1 \in \mathbb{C}$ *be a branch point of the (Galois) cover* $\varphi : X \to \mathbb{P}^1$. *Let* p_1 *be a point of* X *over* a_1, *and let* $g \in Aut(X/\mathbb{P}^1)$ *be the distinguished generator of the inertial group of* p_1. *Let* e *be the order of* g. *Let* $\beta \in Aut(\mathbb{C})$, *and let* m *be an integer such that* β^{-1} *acts on the* e*-th roots of unity as* $\eta \mapsto \eta^m$. *Then* $(g^\beta)^m$ *is the distinguished generator of the inertial group of* p_1^β.

Proof: Let $\mathcal{O}_1$ be the local ring of X at p_1. Its completion contains an element t with $t^e = x$, where x is the image of $\varphi - a_1$ in $\mathcal{O}_1$. ¿From the definition of distinguished inertial group generator, it follows that $g(t) = \eta_e t$, where $\eta_e = \exp(2\pi i/e)$ is the "distinguished" primitive e-th root of 1. Then $g^\beta(t^\beta) = \eta_e^\beta t^\beta$, hence $(g^\beta)^m(t^\beta) = \eta_e t^\beta$. This proves the claim. ∎

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